NASA TECHNICAL TRANSLATION

A NEW METHOD OF APPROACHING THE DESIGN OF SOUNDING ROCKETS

by E. Gismondi

A Report of the International Telecommunications Institute,
Presented at the XIII International Telecommunications Meeting,
12-16 October 1965, *Geneva*

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • JULY 1967

A NEW METHOD OF APPROACHING THE

DESIGN OF SOUNDING ROCKETS

By E. Gismondi

Translation of "Su un Nuovo Metodo per l'Impostazione del Progetto dei Missili Sonda." A Report of the International Telecommunications Institute, Presented at the XIII International Telecommunications Meeting, 12-16 October, 1965, Geneva.

SYMBOLS

v	= velocity of the rocket	m/sec
v_c	= velocity of the missile at burnout	m/sec
t	= time	sec
t_c	= duration of combustion	sec
v_e	= effective velocity of exhaust gas	m/sec
W	= weight of rocket	kg
W_0	= weight of rocket at launch	kg
W_u	= weight of payload	kg
W_p	= weight of propellant	kg
W_s	= weight of structure	kg
W_c	= weight of rocket at burnout	kg
G	= load bearing weight of exhaust gas	kg/sec
R	= aerodynamic resistance of rocket	kg
C_R	= f (M) coefficient of aerodynamic resistance	
$\bar{C}_R$	= f (V_R) coefficient of resistance in propelled flight	
$\bar{C}_{RC}$	= coefficient of resistance in inertial flight	
g	= acceleration of gravity	m/sec^2
S	= propellant decay	kg
S_m	= main section of rocket reference	m^2
ρ	= density of the air	$\dfrac{\text{kg sec}^2}{\text{m}^4}$
$\bar{\rho}$	= average density in propelled flight	$\dfrac{\text{kg sec}^2}{\text{m}^4}$
ρf	= average density in inertial flight	$\dfrac{\text{kg sec}^2}{\text{m}^4}$
α	= ratio of the density	m
Z	= altitude of the rocket	m
Z_c	= altitude at burnout	
Z_f	= maximum altitude attained by the rocket	m
ϵ	= dimensionless quantity (10)	
ϵ_c	= dimensionless quantity calculated by $t = t_c$	
τ	= dimensionless quantity (11)	
σ_2	= dimensionless quantity (12)	
A	= dimensionless quantity (21)	

FOLD LINE

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, D.C. 20546

OFFICIAL BUSINESS

POSTAGE AND FEES PAID
NATIONAL AERONAUTICS & SPACE ADMINISTRATION

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
CODE USS-T
WASHINGTON, D.C. 20546

NASA TTF No.
489

FOLD LINE

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
TECHNICAL TRANSLATION EVALUATION

TO:
THE USERS OF THIS TRANSLATION ⟶

NASA TTF NO.
489

MAINTAINING THE QUALITY OF NASA TRANSLATIONS REQUIRES A CONTINUOUS EVALUATION PROGRAM. PLEASE COMPLETE AND MAIL THIS FORM TO AID IN THE EVALUATION OF THE USEFULNESS AND QUALITY OF THE TRANSLATING SERVICE.

THIS PUBLICATION (*Check one or more*)

☐ FURNISHED VALUABLE NEW DATA OR A NEW APPROACH TO RESEARCH.

☐ VERIFIED INFORMATION AVAILABLE FROM OTHER SOURCES.

☐ FURNISHED INTERESTING BACKGROUND INFORMATION.

☐ OTHER (*Explain*): _____

FOLD LINE — FOLD LINE

TRANSLATION TEXT (*Check one*)

☐ IS TECHNICALLY ACCURATE.

☐ IS SUFFICIENTLY ACCURATE FOR OUR PURPOSE.

☐ IS SATISFACTORY, BUT CONTAINS MINOR ERRORS.

☐ IS UNSATISFACTORY BECAUSE OF (*Check one or more*):

☐ POOR TERMINOLOGY.   ☐ NUMERICAL INACCURACIES.
☐ INCOMPLETE TRANSLATION. ☐ ILLEGIBLE SYMBOLS, TABULATIONS, OR CURVES.

☐ OTHER (*Explain*):
_____

FOLD LINE — FOLD LINE

REMARKS

FROM   DATE

NOTE: REMOVE THIS SHEET FROM THE PUBLICATION, FOLD AS INDICATED, STAPLE OR TAPE, AND MAIL.
NO POSTAGE NECESSARY.

NASA Form 1332   AUG 66   GPO 915-777

CUT ALONG THIS LINE

CUT ALONG THIS LINE

A NEW METHOD OF APPROACHING
THE DESIGN OF SOUNDING ROCKETS

E. Gismondi, Doctor of Engineering

ABSTRACT. The author of this report intends to present rocket
designers a very simple calculating method of precisely deter-
mining the principal characteristics of a rocket that meets
the requirements set forth by the design problem. The method
may also be used to compare the influence of some parameters on
the characteristics and the performance of rockets. The method
proposed has been applied to analytical solutions which require
only a certain amount of simplification and which can be elimi-
nated with iteration.

INTRODUCTION

In general, in order to set up a design project for a single-stage rocket   /3*
in vertical ascent, tests are made based on similar rockets and on personal
experience, with the aid of a digital computer.

This report is intended to give designers a very simple calculating method
of precisely determining the principal characteristics of a rocket that meets
the requirements set forth by the design problem. This method can also be
used to compare the influence of some parameters on the characteristics and
performance of rockets.

With successive approximations, it is possible to approach results, which
are satisfactory for setting up a design project.

The introduction of nondimensional quantities resulted in a nondimensional
solution, with respect to selecting from the family of rockets that one which
might satisfy the requirements.

The new method has been applied to analytic solutions, which only require
a certain amount of simplification, and which can be eliminated with the
iteration of the method application.

With the attached nondimensional graphs, however, a digital computer was
used for greater precision. At the same time, the principle of the solution
continued to be valid.

1. Propelled Ascent   /4

To calculate the velocity and altitude attainable by a vertically launched,
single-stage sounding missile, it is necessary to integrate the differential

---

* Numbers in the margin indicate pagination in the foreign text.

equation valid for the propelled portion of the rocket's flight:

$$\frac{dv}{dt} = v_e \frac{G}{W} - g - \frac{R_g}{W}$$ (1)

To simplify the calculation process without detracting from the principle of the problem, the rocket was considered at angle of trim 0 in regard to the trajectory, then aerodynamic lift 0, in an unperturbed atmosphere.

The terms of Equation (1) are as follows:

$$dv = v_e \frac{G}{W_0 - Gt} dt - gdt - \frac{C_R \ 1/2 \ \rho \ S_m \ v^2 g}{W_0 - Gt} dt$$ (2)

It is not possible to integrate Equation (2) analytically since the drag coefficient C_r is a function of the Mach number of flight, the density of the air depends on the altitude, the effective velocity of the exhaust gas depends on the ambient pressure (and thus on the altitude), and the acceleration of gravity g depends on the altitude.

For the approximate analytic integration of Equation (2), some simplified hypotheses are presented:

$v_e = \bar{v}_e$ = constant (3): the error which can be made in calculating the thrust is 10% at the most; however, in an iteration of the calculation method, it is possible to adopt a more realistic average value.

$g = \bar{g}$ = constant (4): the error made is negligible.

$\rho = \bar{\rho}$ = constant (5): an average value of the density in the propelled portion of the rocket's flight is assumed, and this value is modified during the iteration of the calculation.

$C_R = \bar{C}_R = f(V_R)$ (6): the velocity of sound is assumed to be constant with the altitude so that the drag coefficient depends only on the velocity. The <u>/5</u> error which is committed is not great since the velocity of sound depends on the square root of absolute temperature $a = \sqrt{K \ RT}$.

$v = v_R$ (7): velocity v which introduces the second part of Equation (2) for the calculation of the drag, is calculated as if the drag were 0, according to the equation:

$$dv = v_e \frac{G}{W_0 - Gt} dt - gdt$$ (8)

from which, integrating between initial instant t = 0 and t, the following is obtained:

$$v_R = - \left[v_e \log \left(1 - \frac{Gt}{W_0} \right) + gt \right]$$ (9)

Velocity v_R is then greater than the effective velocity of flight at the same instant t.

The drag calculated with v_R is greater than the effective resistance.

In Equation (2) an error is introduced which causes the velocity and the altitude attainable by the rocket to be less than real values.

The following nondimensional variables are given:

$$\epsilon = \frac{Gt}{W_0} \qquad\qquad \epsilon_c = \frac{Gt_c}{W_0} \tag{10}$$

$$\tau = \frac{\overline{\rho S_m} v_e^2}{G/g\ v_e} \tag{11}$$

$$\sigma = \frac{G/g\ v_e}{W_0} \tag{12}$$

For a discussion of the significance of the dimensionless quantities, see Paragraph 3.

Equation (9) is transformed as follows:

$$v_R = -v_e \left[\log\ (1-\epsilon) + \frac{\epsilon}{\sigma} \right] \tag{13}$$

Since the drag coefficient $\overline{C}_R = f(v_R)$ depends only on v_R , the following is obtained:

$$\overline{C}_R = f(\epsilon,\sigma) \tag{14}$$

By substituting positions (3), (4), (5), (7), (10), (11), (12), and (14) in Equation (2) integration is obtained between $= 0$ and $= c = \dfrac{G\ t_c}{W_0}$

to obtain v'_c.

This velocity will be less than the real value since it achieved, in the calculation of the drags, a velocity v_R greater than the real value. To introduce this error, burnout velocity is assumed to be the average value:

3

$$v_c = \frac{v_c{}^! + v_R}{2} \tag{15}$$

between the velocity calculated with a resistance higher than the real value, and the velocity calculated without resistance:

$$v_c = \overline{v}_e \left[-\log (1-\epsilon_c) - \frac{\epsilon_c}{\sigma} - \frac{\tau}{4} \int_0^{\epsilon_c} \frac{\overline{C}_R (\epsilon,\sigma) [\log (1-\epsilon) + \frac{\epsilon}{\sigma}]^2}{1 - \epsilon} d\epsilon \right] \tag{16}$$

Integrating again, the altitude attainable by the rocket at burnout is obtained:

$$z_c = \frac{\overline{v}_e{}^2}{J} \left[\frac{1-\epsilon_c}{\sigma}\log(1-\epsilon_c) + \frac{\epsilon_c}{\sigma} \frac{\epsilon_c{}^2}{2\sigma^2} - \frac{\tau}{4\sigma} \int_0^{\epsilon_c}\int_0^{\epsilon} \frac{\overline{C}_R [\log (1-\epsilon) + \frac{\epsilon}{\sigma}]^2}{1 - \epsilon} d\epsilon \, d\epsilon \right] \tag{17}$$

2. Inertial Ascent

The vertical ascent of the rocket, after the propellant is consumed, contin- navigationues through inertia, according to the differential equation:

$$\frac{dv}{dt} = -g - \frac{Rg}{W_0} = - g - \frac{C_R \, 1/2 \, \rho_g \, S_m \, v^2 \, g}{W_0 \, (1 - \frac{W_p}{W_0})} \tag{18}$$

It is not possible to integrate this equation analytically, due to the same reasons stated for Equation (2).

The following simplified hypotheses are given:

$g = \overline{g}$ = constant (4): the error committed is still negligible.

$\rho_g = \alpha\overline{\rho}$ = constant (19). A constant value of the density proportional to value $\overline{\rho}$, adopted in the propelled portion of the rocket's flight, is assumed. It is possible to reduce the error with successive iterations of the calculation.

$C_R = \overline{C}_{RC}$ = constant (20): a constant value of the drag coefficient is assumed, since a fair proportion of the inertial flight is achieved at supersonic velocity.

The dimensionless variable is introduced:

$$A^2 = \frac{2(1-\epsilon_c)}{\overline{C}_{RC} \, \alpha\tau\sigma} \tag{21}$$

By introducing dimensionless variables (10), (11), (12), and (21), and positions (4), (19), and (20), Equation (18) becomes:

$$dv = -g \left[1 + \frac{1}{A^2} \left(\frac{v}{v_e} \right)^2 \right] dt \tag{22}$$

Equation (22) is integrated with the following boundary conditions:

$$\left. \begin{array}{l} t = t_c \\ v = v_c \\ z = z_c \end{array} \right\} \quad \text{initial} \tag{23}$$

$$\left. v = 0 \; \right\} \quad \text{final} \tag{24}$$

$$v = \overline{v}_e \, A \, \tan \left[(t_c - t) \frac{\overline{g}}{A \, \overline{v}_e} + \text{arc tan} \left(\frac{1}{A} \frac{v_c}{\overline{v}_e} \right) \right] \tag{25} \quad \underline{/8}$$

$$t_f - t_c = \frac{\overline{v}_e \, A}{\overline{g}} \, \text{arc tan} \left(\frac{1}{A} \frac{v_c}{\overline{v}_e} \right) \tag{26}$$

$$z_f - z_c = - \frac{\overline{v}_e^2}{\overline{g}} \, A^2 \, \log \cos \, \text{arc tan} \left(\frac{1}{A} \frac{v_c}{\overline{v}_e} \right) \tag{27}$$

3. Discussion and Significance of the Nondimensional Quantities

All of the nondimensional quantities selected have physical significance:

$$\varepsilon_c = \frac{G \, t_c}{W_o} = \frac{W_p}{W_o}$$

(10) represents the percentage of propellant with respect to the total initial weight of the missile.

$$\tau = \frac{S_m \, \overline{v}_e^2 \, \overline{\rho}}{G/\overline{g} \, \overline{v}_e}$$

(11) represents the main thrust section ratio.

$$\sigma = \frac{\frac{G}{\overline{g}} \, \overline{v}_e}{W_o} = \frac{s}{W_o}$$

(12) represents the acceleration at launching

For a family of rockets, the ratio between the weight of the structure and the propellant, and the weight of the propellant remains constant.

$$\gamma = \frac{w_s}{w_p} \tag{28}$$

5

It is possible to give the following definition of:

$$\frac{w_u}{w_0} = 1 - \varepsilon_c (1 + \gamma),$$ (29)

the ratio between the payload and the weight of the rocket at launch.

It is convenient to seek the minimum value of ε_c in order to have the maximum payload at equal weight at the time of launch.

$$\frac{1}{A^2} = \frac{\overline{C}_{RC} \frac{1}{2} S_m \rho_f \overline{v}_e^2}{w_s + w_u} = \frac{R_f}{w_c}$$ (21)

represents the ratio between the hypothetical aerodynamic resistance of the rocket in inertial flight, calculated with the velocity $v = \overline{v}_e$, and the weight of the rocket at burnout. Thus parameter $1/A^2$ represents the hypothetical deceleration of the rocket in inertial flight and hence an estimate of the aerodynamic braking at thrust decay.

4. Method for Setting Up the Design Project.

For the design of a single-stage sounding rocket, i.e., to determine the principal characteristics for the required performance, it is necessary to establish in advance:

- the aerodynamic shape and thus the behavior of the coefficient of resistance (6) and (20);
- the type of propellant and thus the v_e (3);
- the propellant system and the related load bearing structure necessary; their weight can be considered proportional to the weight of the propellant

$$\gamma = \frac{W_s}{W_p} \quad (28);$$

- the distribution of the density of the air and thus $\overline{\rho}$ (5) and $\alpha \overline{\rho}$ (19);
- the average value of the acceleration of gravity $\overline{g}$ (4).

On the other hand, the following are known:

$$w_0 = w_u + w_s + w_p \qquad \text{total initial weight} \qquad (30)$$

$$w_p = G t_c \qquad \text{initial propellant weight} \qquad (31)$$

$$s = \frac{G}{\overline{g}} \overline{v}_e \qquad \text{constant thrust} \qquad (32)$$

6

The variables in question are G, t_c, S, S_m, W_0, W_u, W_p, W_s, while there are only 7 relations which define the problem. By using Equations (28), (30), (31), and (32), it would be possible to reduce the problem to four variables defined by three relations. But by way of comparison, we might state that it is not convenient to proceed this way. In fact, the variable that can be imposed on the problem and which can then be used to define analytically all the other quantities, can also be one of the four variables defined by Equations (28), (30), (31), and (32). In this way, the designer has greater liberty in defining the rocket, (or rather the family of rockets) which can satisfy the requirements of the design project itself.

By adopting appropriate criteria, suggested from time to time by the problem posed, it is possible by means of Equation (27), which connects the dimensionless quantities ε_c τ σ among themselves, to calculate the value of the quantities themselves (as seen in the graph of Figure 5).

The above four unknown variables, interconnected by definitions (10), (11), and (12) of dimensionless quantities ε_c τ σ, characterize a family of similar rockets. Since one of the four variables is fixed by the data of the problem, the remaining unknown variables prove to be dimensional. In this way, the assumed dimensional quantity permits the selection of a well defined rocket to be made from similar rockets, thus satisfying the requirements of the problem.

The exactness of the method depends on the values assigned to the constants and the error introduced in calculating the aerodynamic resistance.

Constants $\bar{g}$ (4) and $\bar{v}_e$ (3) can be substituted, after an initial calculation, by values closer to reality. The average value of density $\bar{\rho}$ (5) in the propelled portion of the rocket's flight can be introduced in expression τ (12), after which altitude Z_c is calculated. Inaccuracy can thus be reduced to a minimum.

The value of ratio α , between the density of the air in the propelled flight portion and in the inertial flight portion, can be substituted in a second estimate, as seen in Paragraph 5, based on the results of the first /11 calculation.

With regard to the error introduced by v_R and hence the resistance, it should be noted that it is possible to eliminate this error completely by using a digital computer for integrating the nondimensional Equations (2) and (18).

Graphs such as those mentioned in Paragraph 5 permit a general solution, depending only on the coefficient of resistance C_R and on ratio α , since all other constants are introduced by nondimensional quantities ε_c τ σ (10), (11), and (12).

The graphs were obtained by a digital computer, with the coefficient of resistance $\overline{C}_R$ and $\overline{C}_{RC}$ = 0.32 and with α = 0.07.

5. Numerical Application of the Method

Given the attainable altitude Z_f and the weight of the payload W_u, the characteristics of the single-stage rocket of minimum initial weight W_0 must be defined.

In order to specify the solution it is first necessary to define the field of possible existence of two nondimensional quantities τ and σ. This is indispensable because of the possibility of limiting the numerous solutions permitted by the ballistic treatment, so as to avoid technically unfeasible solutions.

Such limitations, well known to rocket technicians and designers, are dictated by practice and can be modified later on the basis of technological evolution.

From the expressions:

$$\frac{v_c}{\overline{v}_e} = F' \ (\varepsilon_c, \ \tau, \ \sigma, \ \overline{C}_R) \tag{16}$$

$$\frac{Z_c}{\overline{v}_e^2 \Big/ \overline{g}} = F'' \ (\varepsilon_c, \ \tau, \ \sigma, \ \overline{C}_R) \tag{17}$$

$$\frac{Z_f}{\overline{v}_e^2 \Big/ \overline{g}} = F''' \ (\varepsilon_c, \ \tau, \ \sigma, \ \overline{C}_R, \ \overline{C}_{RC}, \ \dot{\alpha}) \tag{27}$$

it is possible to graph the maximum altitude attainable by the rocket Z_f, as a /12 function of the nondimensional quantities $\varepsilon_c \ \tau \ \sigma$ (as seen in Figures 1, 2, 3, and 4).

In the Figure 5 graph, the situation is represented for the projected altitude Z_f, which is a factor in the problem, and for the possible values of $\varepsilon_c \ \tau \ \sigma$.

We select the pair of values τ and σ which minimize ε_c in that by

$$\frac{W_u}{W_0} = 1 - \varepsilon_c \ (1 + \gamma) \tag{35}$$

8

the minimum value of W_0 is obtained for the minimum value of ε_c.

Knowing $\varepsilon_c \ \tau \ \sigma \ W_0 \ W_u$, it is possible to calculate from (28), (29), (31), and (32), all of the other quantities which are characteristic of the rocket in the first approximation.

In a second approximation, it is possible to improve the values calculated, while bearing in mind a more realistic distribution of the density of the air.

Z is calculated on the basis of (21), or on the basis of the graphs reported in Figures 1, 2, 3, and 4 and the average value is determined for density ρ at the end of altitude Z_c (rocket in propelled flight) and at the end of altitude Z_ρ (rocket in inertial flight). We may then determine the value:

$$\alpha' = \frac{\bar{\rho}}{\rho_f}$$

We calculate $v_c/\bar{v}_e$ on the basis of Equation (20) or on the basis of the graphs in Figures 6, 7, 8, and 9.

From Equation (27), or the graphs in Figure 10, we learn for:

$$(Z_f - Z_c)\Big/\bar{v}_e^2\Big/\bar{g} \text{ and } v_c\Big/\bar{v}_e \text{ is the value of A.}$$

In expression (21) of the nondimensional quantities:

$$A = \sqrt{\frac{2(1-\varepsilon_c)}{\bar{C}_{RC}\alpha \ \tau \ \sigma}} \quad \text{we substitute the new, calculated value}$$

of α'. While τ remains the same, we must again seek values for ε_c and σ that satisfy Equation (21) and, at the same time, Equation (27), which can be represented by the graph of Figure 5 b.

To find such pairs of values (knowing A, $\bar{C}_{RC}, \alpha', \ \tau$), the line representing Equation (21) is seen in Figure 5.

$$\sigma = \frac{2}{A^2 \ \bar{C}_{RC}\alpha' \tau} - \varepsilon_c \frac{2}{A^2 \ \bar{C}_{RC}\alpha' \tau} \tag{33}$$

9

The point of intersection with the graph parametered at τ, represents the values of ε_c and σ.

Thus we obtain a new rocket, with characteristics and design approximating a real rocket.

Numerical Example

A rocket with the following characteristics is considered:

$$Z_f = 160\ 000\ m$$
$$W_u = 45\ kg$$
$$\overline{V}_e = 1930\ m/sec$$
$$\gamma = 0.40$$

It is assumed that $g = g$, $C_{RC} = 0.32$, $\alpha = 0.07$.

From the graphs in Figures 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10, we obtain

for $\dfrac{Z_f}{\overline{v}_e^2/\overline{g}} = 0.42$ the pair of values $\sigma = 12$ and $\tau = 2$, which minimizes $\varepsilon_c = 0.685$.

Furthermore, we obtain $\dfrac{v_c\sqrt{\ }}{v\sqrt{\ }} = 0.98$, $\varepsilon_c\sqrt{\dfrac{\overline{v}_e^2}{\overline{g}}} = 0.10$, $\overline{\rho} = 5.9 \cdot 10^{-2}$, $\rho_f = 8.0 \cdot 10^{-6}$, $\alpha' = 1.35 \cdot 10^{-4}$, and $A = 26$.

Equation (21) thus expresses the straight line:

$$\sigma = 34.4 - \varepsilon_c \cdot 34.4$$

which is traced on the graph of Figure 5b.

The point of intersection: $\sigma' = 10.2$, $\tau' = 2$, $\varepsilon_c' = 0.690$, $Z_f' = 162\ 000\ m$, $Z_c' = 41\ 500\ m$, $v_c = 1870\ m/sec$, $W_0 = 1300\ kg$, $W_p = 896\ kg$, $W_s = 359\ kg$, $t_c = 13.1\ sec$, $G = 68.5\ kg/sec$, $S = 13\ 300\ kg$, $S_m = 0.125\ m^2$.

6. Numerical Application of the Method

Given the initial weight of rocket W_0, determine the types of rockets having the maximum possible altitude Z_f and the maximum payload W_u.

Assuming that the following values are known: $\overline{v}_e$, γ, $\overline{C}_{RC}$, $\overline{C}_R$, α

we sketch, with the help of Equation (27), graphs ε_c = constant, which gives the altitude:

$$Z_f \Big/ \overline{v}_e^{\,2} \Big/ \overline{g}$$

as a function of σ and parametered at τ. See Figures 11, 12, 13, and 14.

The payload W_u is determined, and from Equation (29) we obtain ε_c. The maximum altitude attainable, and the values of two nondimensional parameters τ and σ, are obtained from the graph corresponding to value ε_c.

This procedure is used for the selected values of the payload and then reported in Table $Z_f = Z_f(W_u)$.

Numerical Example

The following values are given:

$$W_0 = 4000 \text{ kg}$$
$$v_e = 2500 \text{ m/sec}$$
$$\rho = 0.29$$
$$C_{RC} = 0.32$$
$$\alpha = 0.07$$

$\dfrac{Z_f}{v_e^{\,2}/g}$	Z_f [m]	W_u [kg]	ε_c	τ	σ	W_p [kg]	W_s [kg]
0.472	295 000	390	0.700	2	6	2800	810
0.392	244 000	648	0.650	2	7	2600	752
0.354	222 000	775	0.625	2	7	2500	725
0.318	198 000	905	0,600	2	8	2400	695

Numerical Application of the Method

Given the completely defined single-stage rocket, determine the payload W_u which increases the rocket's attainable altitude Z_f.

From definitions (10) and (12) of the nondimensional quantities ε_c and σ:

$$\sigma = \frac{G\,\overline{v}_e}{\overline{g}\,W_0} = \frac{G\,\overline{v}_e}{\overline{g}\,W_p}\,\varepsilon_c = \sigma_1 \cdot \varepsilon_c \qquad (34)$$

$$\sigma_1 = \frac{G\,\overline{v}_e}{\overline{g}\,W_p} = \text{constant} \qquad (35)$$

Equation (34) is substituted in Equation (27), which then becomes a function of τ, ε_c, and $\sigma = $ constant.

By deriving and equating Equation (27) to zero, which is thus modified, it is possible to determine the value of ε_c which enables the rocket to attain the highest altitude.

Otherwise, it is possible to get the value of ε_c from the graphs of Figures 15, 16, 17, and 18 where, according to Equation (27), as modified, Z_f is traced as a function of σ_1 and τ.

Numerical Example

$$W_p = 34 \text{ kg}$$
$$W_s = 13.6 \text{ kg}$$
$$S_m = 0.0177 \text{ m}^2$$
$$G = 8.1 \text{ kg/sec}$$
$$t_c = 4.2 \text{ sec}$$
$$\overline{v}_e = 1860 \text{ m/sec}$$
$$\rho = 10 \cdot 10^{-2}$$

$$\overline{\tau} = \frac{\overline{\rho}\,S_m\,\overline{v}_e\,\overline{g}}{G} = 4$$

$$\sigma_1 = \frac{G\,\overline{v}_e}{\overline{g}\,W_p} = 45$$

From Figure 16, we obtain $\varepsilon_c = 0.670$, from which:

$$W_0 = 51.7 \text{ kg}$$
$$W_u = 4.1 \text{ kg}$$

12

From Figure 2, we obtain $Z_f = 69\ 000$ m; $Z_c = 3300$ m.

Optimization of the altitude attainable by a rocket defined by factor ε_c which, as we know, is the ratio between the weight of the propellant and the initial weight of the rocket, can be deduced easily by deriving Equation (27) with regard to ε_c and equating it to zero.

$$\frac{d(Z_f/\bar{v}_e^2/\bar{g})}{d\varepsilon_c} = \frac{d(Z_c/\bar{v}_e^2/\bar{g})}{d\varepsilon_c} - \frac{\bar{v}_e^2}{\bar{g}}\frac{J(Z_f-Z_c)}{(A^2)}\frac{d(A^2)}{d\varepsilon_c} - \frac{\bar{v}_e^2}{\bar{g}}\frac{J(Z_f-Z_c)}{\phi\left(\frac{v_c}{\bar{v}_e}\right)}\frac{d\left(\frac{v_c}{\bar{v}_e}\right)}{d\varepsilon_c} \quad \text{(a)}$$

The expressions of the individual terms which make up Equation (a) are explained as follows:

$$\frac{d(Z_c/\bar{v}_e^2/\bar{g})}{d\varepsilon_c} = -\frac{\log(1-\varepsilon_c)}{\sigma} - \frac{1}{\sigma} - \frac{\varepsilon_c}{\sigma^2} - \frac{\tau}{4\sigma}\int_0^{\varepsilon_c}\frac{\bar{C}_R[\log(1-\varepsilon)+\frac{\varepsilon}{\sigma}]^2}{1-\varepsilon}\,d\varepsilon \quad \text{(b)}$$

$$\frac{\phi(Z_f-Z_c)}{\phi(A^2)} = -\log\cos\arctan\frac{1}{A}\frac{v_c}{\bar{v}_e} - \frac{(\frac{1}{A}\frac{v_c}{\bar{v}_e})^2}{1+(\frac{1}{A}\frac{v_c}{\bar{v}_e})^2} \quad \text{(c)}$$

$$\frac{d(A^2)}{d\varepsilon_c} = -\frac{2}{\bar{C}_{RC}}\alpha\tau\sigma \quad \text{(d)}$$

$$\frac{\phi(Z_f-Z_c)}{\phi(\frac{v_c}{\bar{v}_e})} = -\frac{\frac{v_c}{\bar{v}_e}}{1+(\frac{1}{A}\frac{v_c^2}{\bar{v}_e})} \quad \text{(e)}$$

$$\frac{d(\frac{v_c}{\bar{v}_e})}{d\varepsilon_c} = \frac{1}{1-\varepsilon_c} - \frac{1}{\sigma} - \frac{\tau}{4}\frac{\bar{C}_R\left[\log(1-\varepsilon_c)+\frac{\varepsilon_c}{\sigma}\right]^2}{1-\varepsilon_c} \quad \text{(f)}$$

13

By substituting (a) for (b), (c), (d), (e), and (f), an equation is found in which τ and σ are assumed to be parameters, and ε_c an unknown factor.

It is not possible to obtain an analytical expression of ε_c for the diverse pair τ and σ, since Equation (a) is transcendental. However, it would be possible to graphically show the field of existence of τ and σ, for which (a) offers a solution, and which are thus values for which a maximum exists for the function:

$$z_f / \overline{v}_e^{\,2} / \overline{g} = f(\varepsilon_c);$$

nowever this would diminish the exactness of the solution to the problem and would be a useless complication to the method explained above.

Figure 1

Figure 2

16
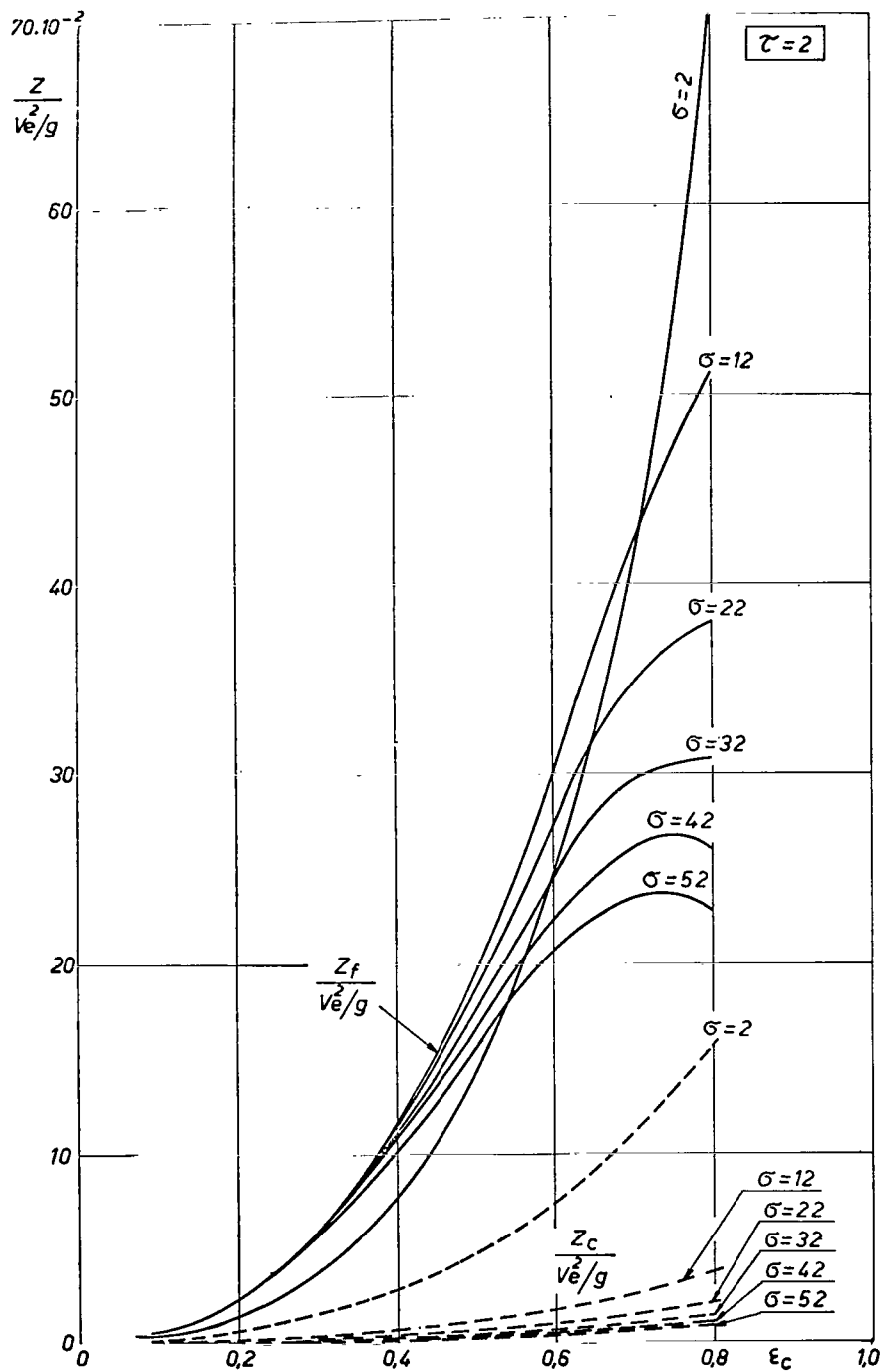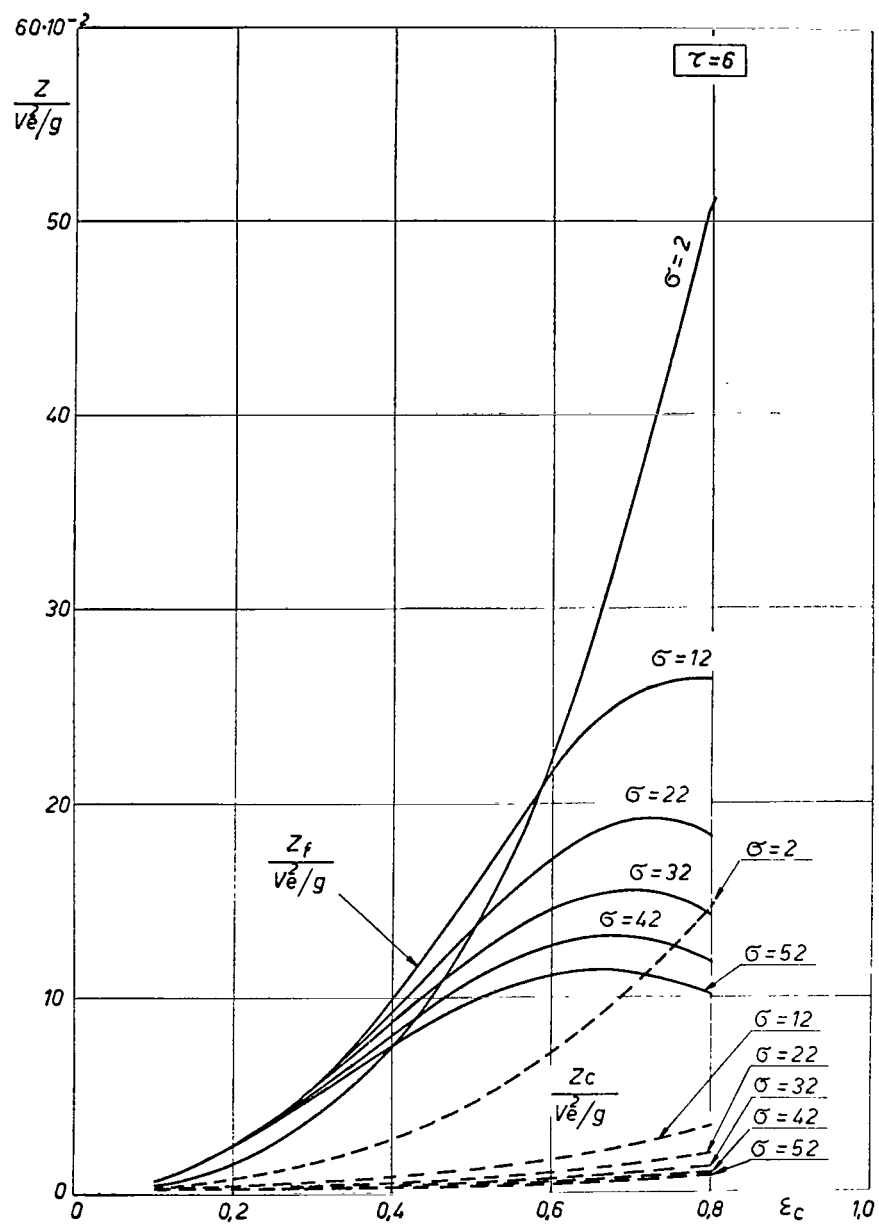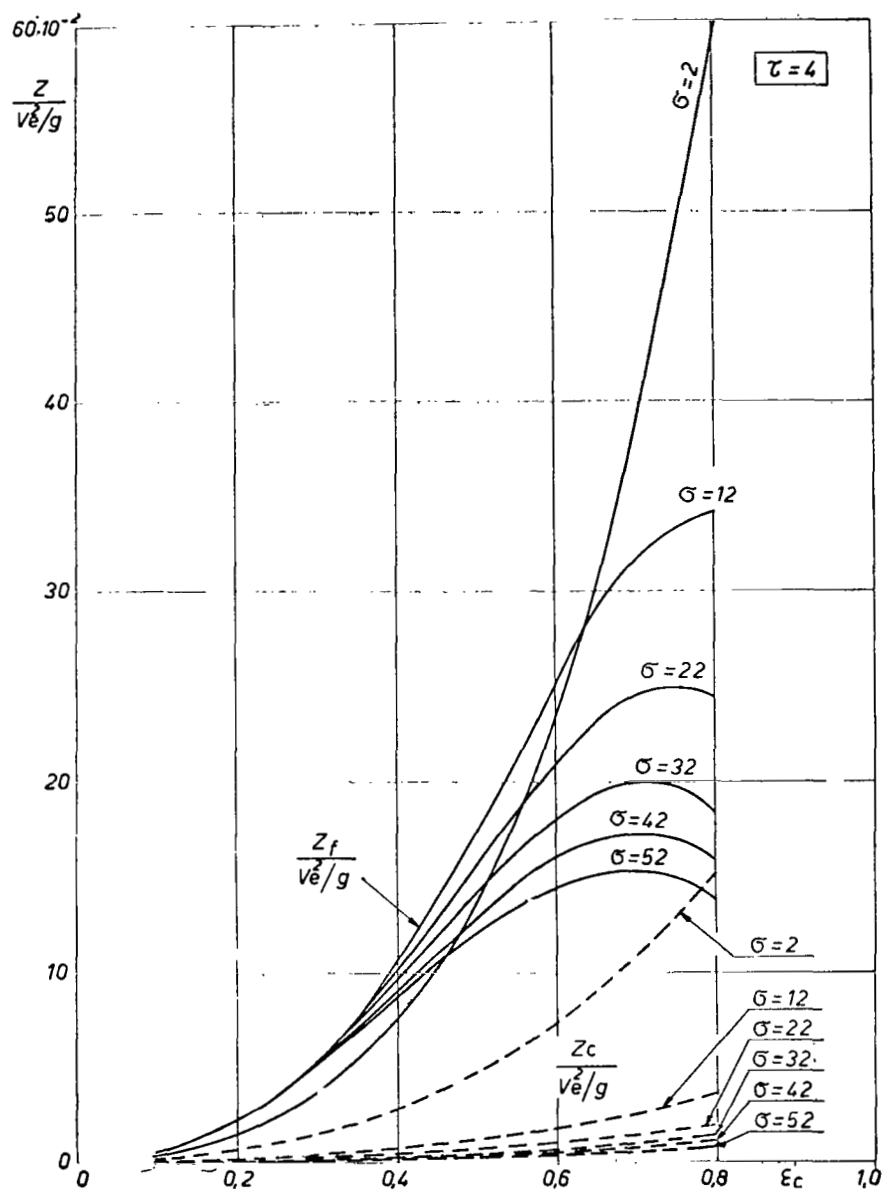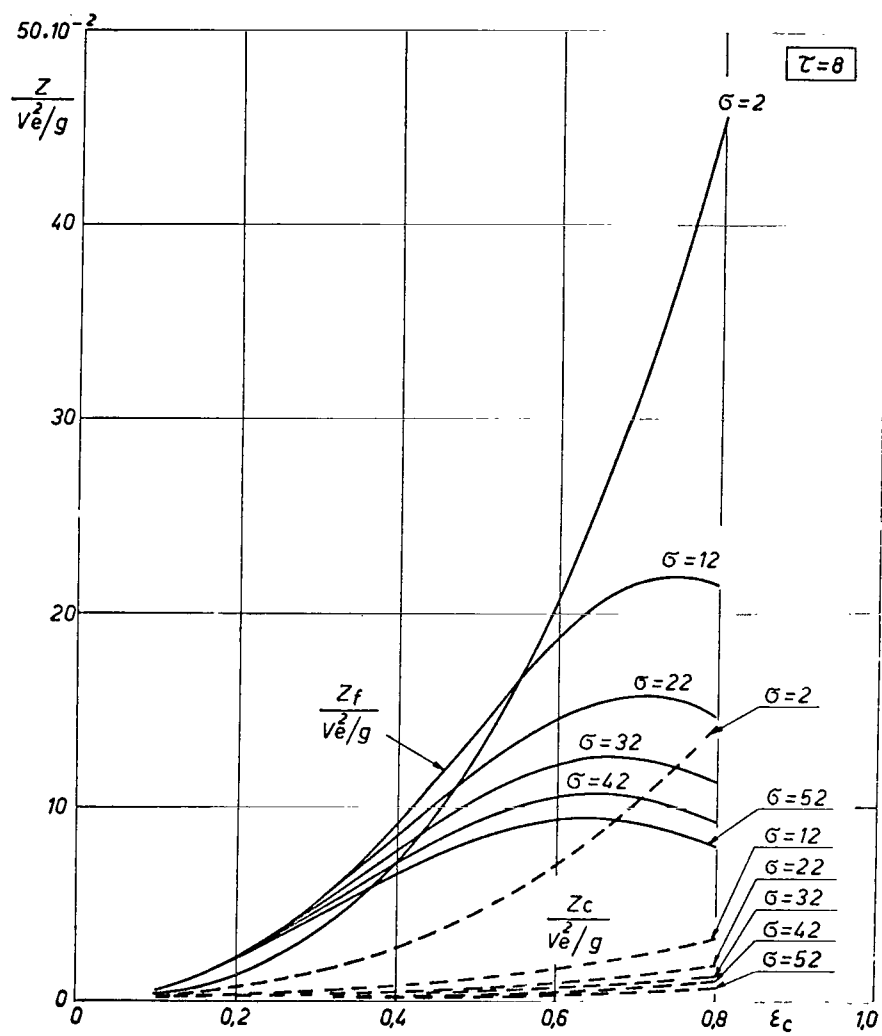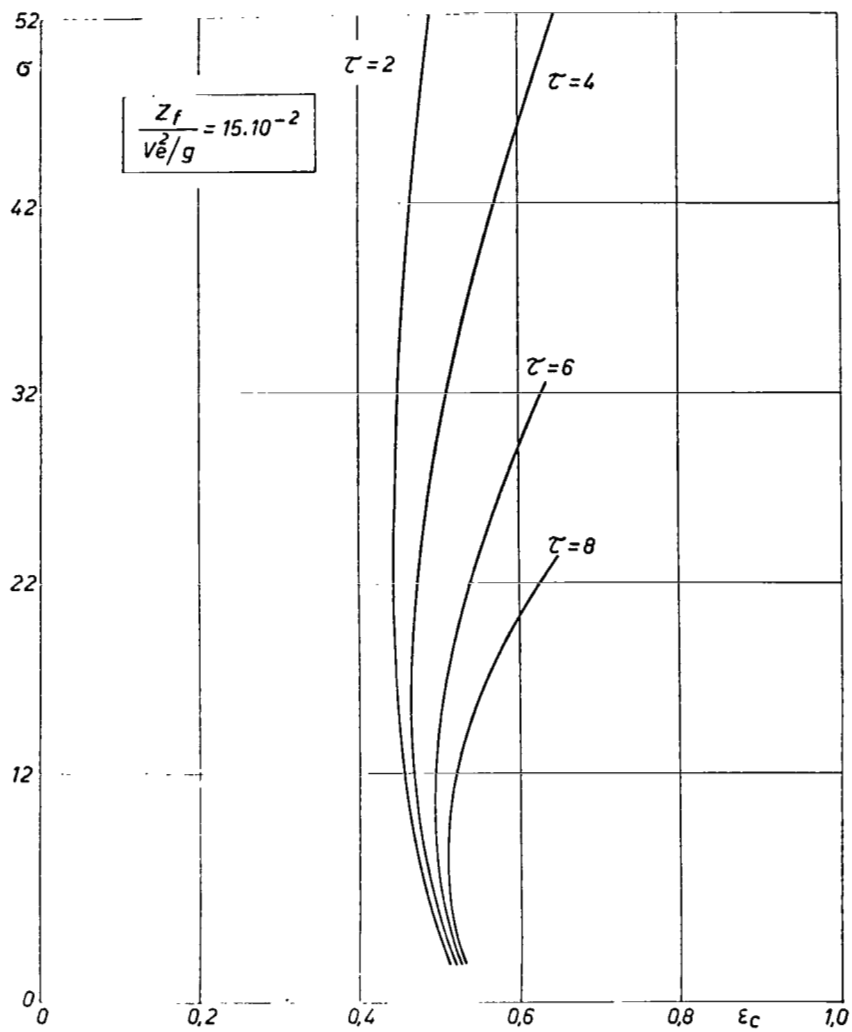
Figure 3

Figure 4

Figure 5
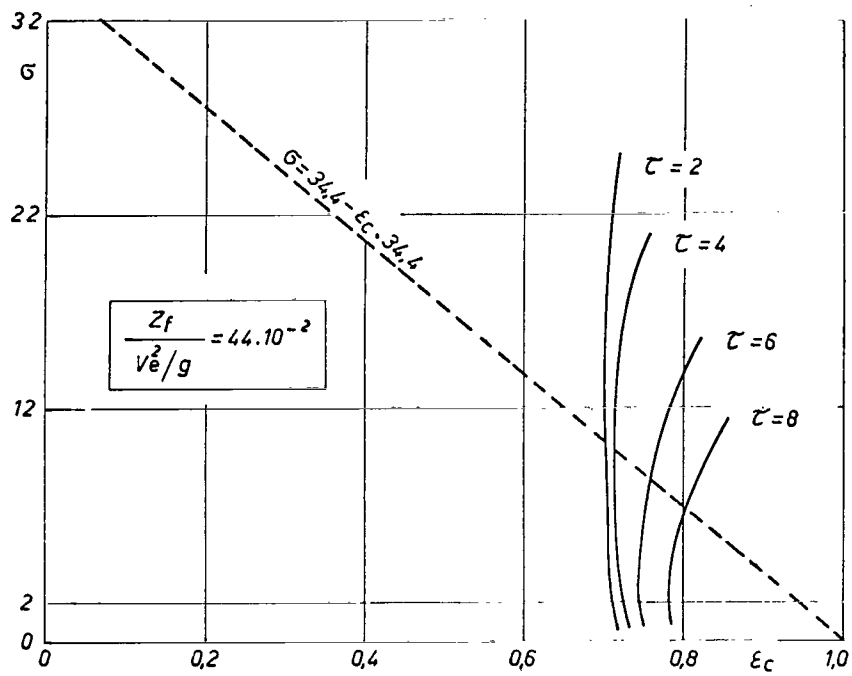
Figure 5b
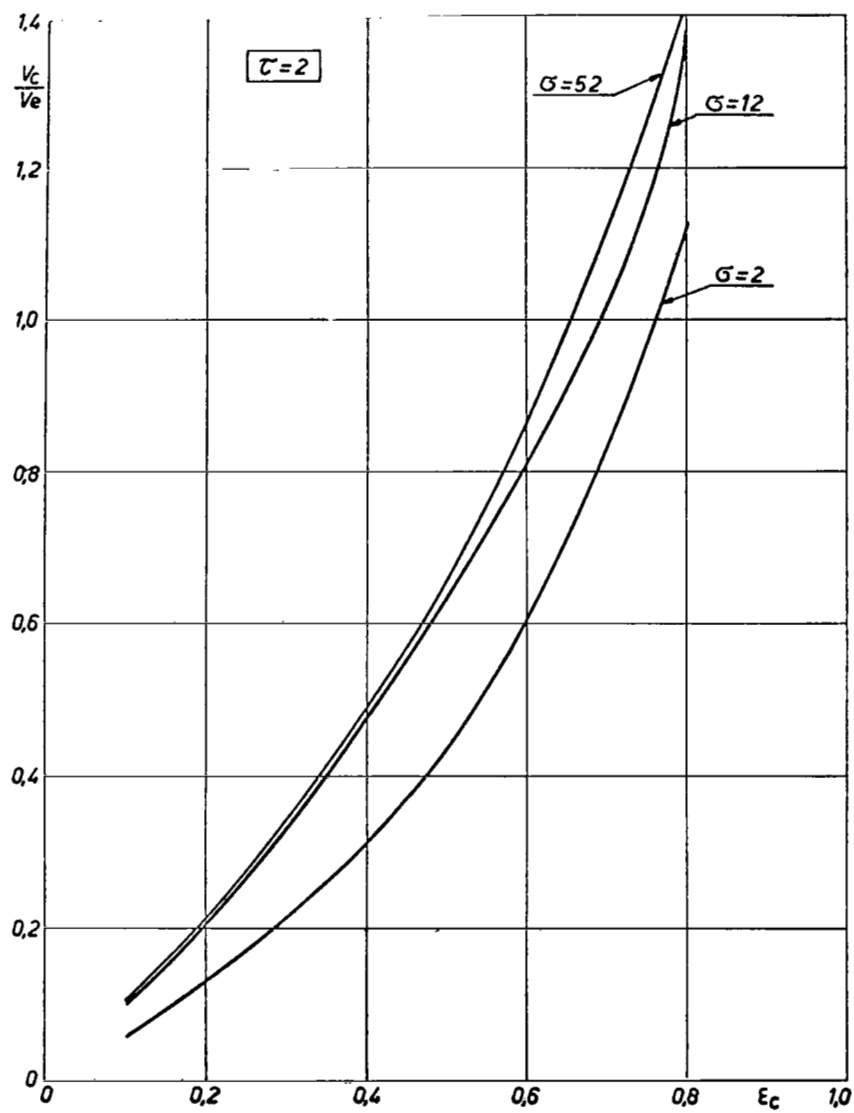
Figure 6
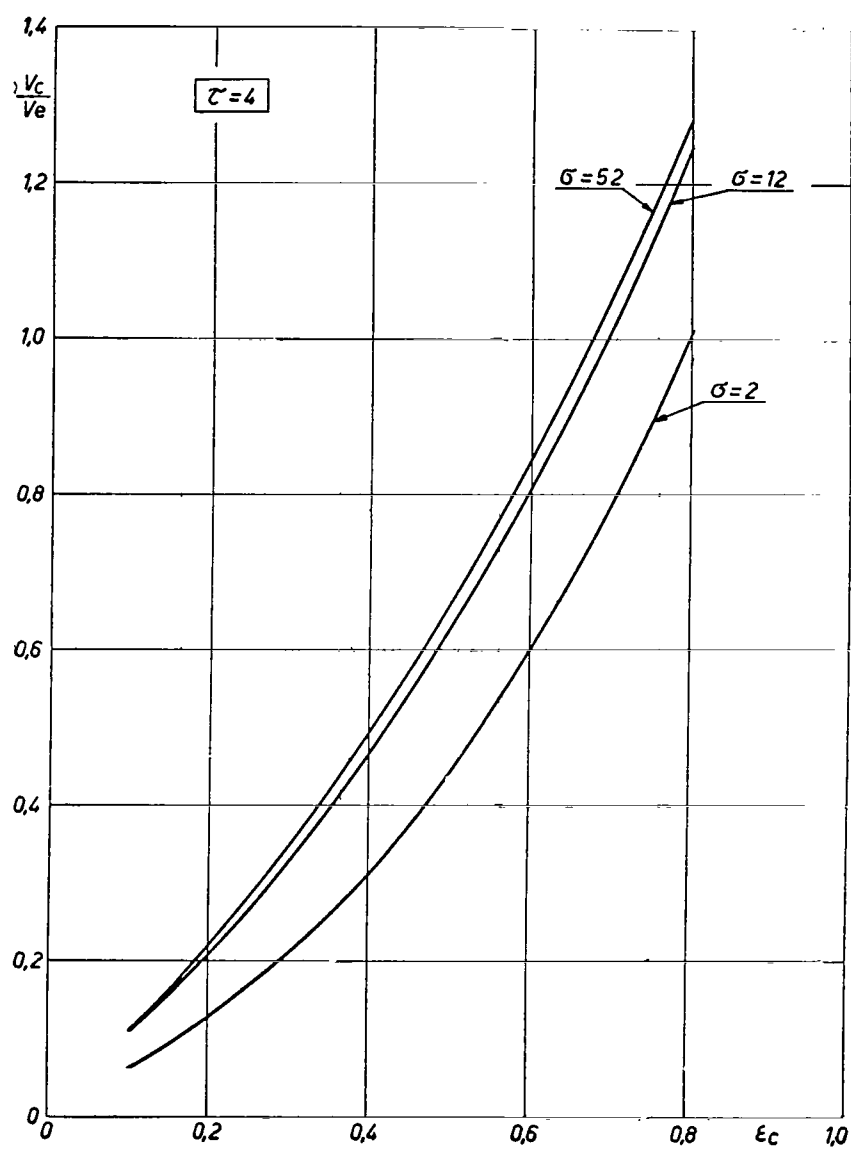
Figure 7
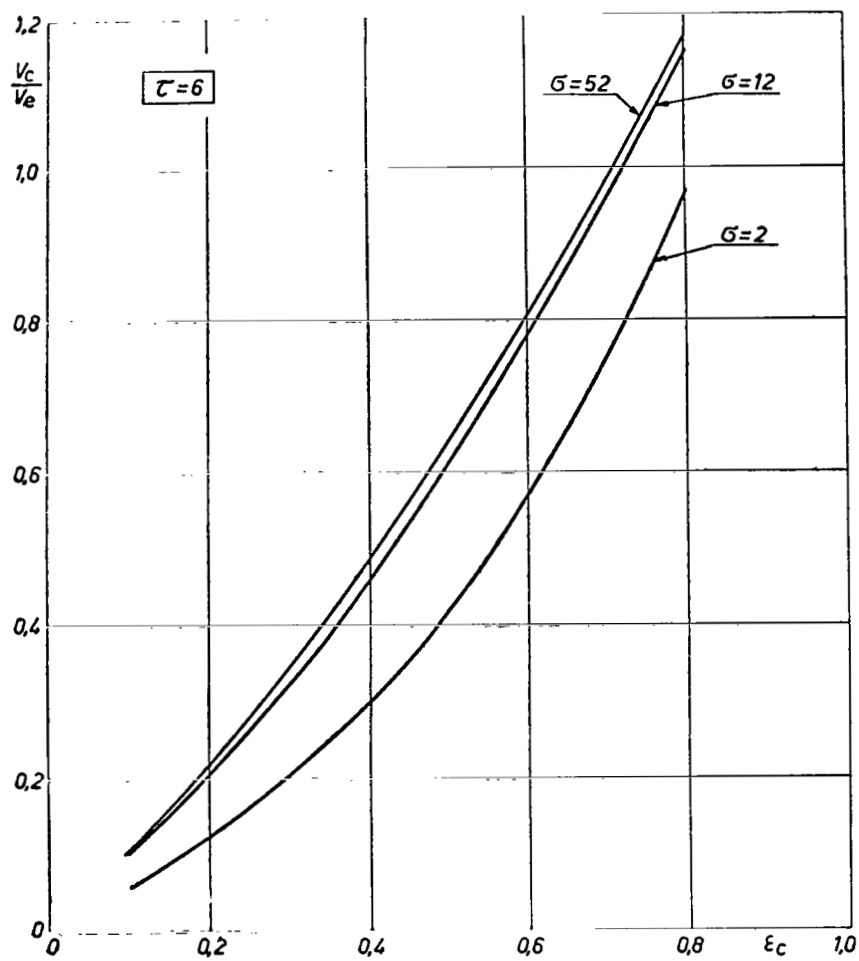
Figure 8
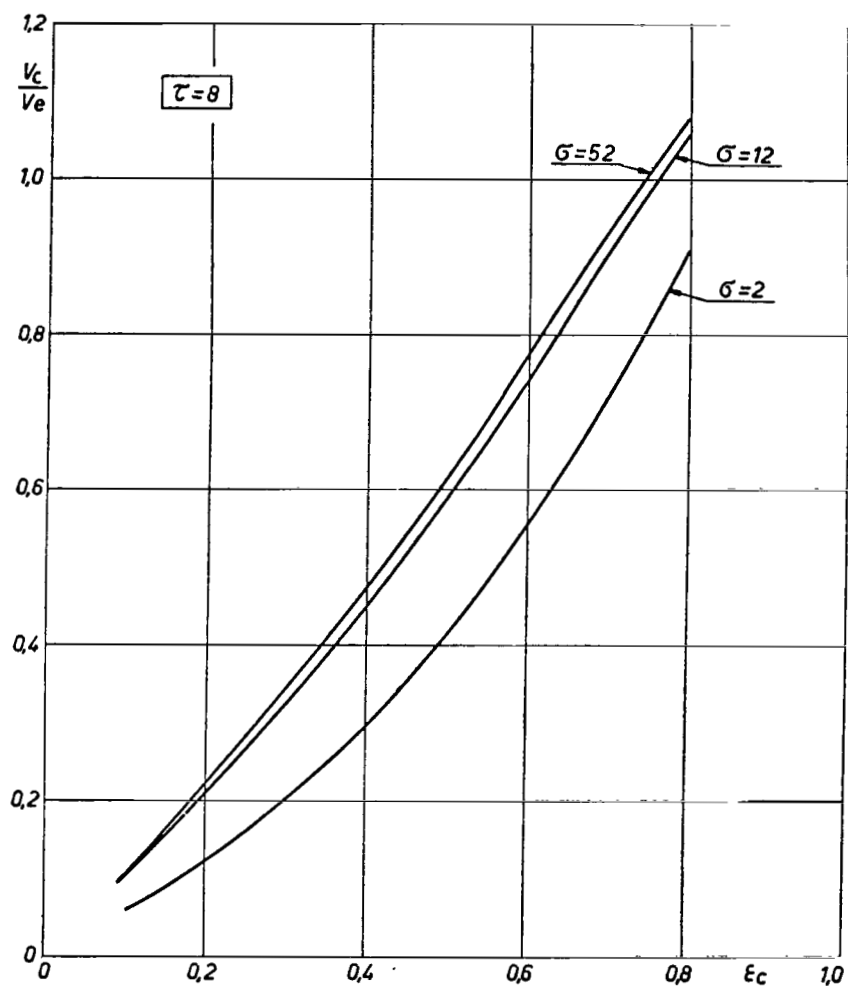
Figure 9
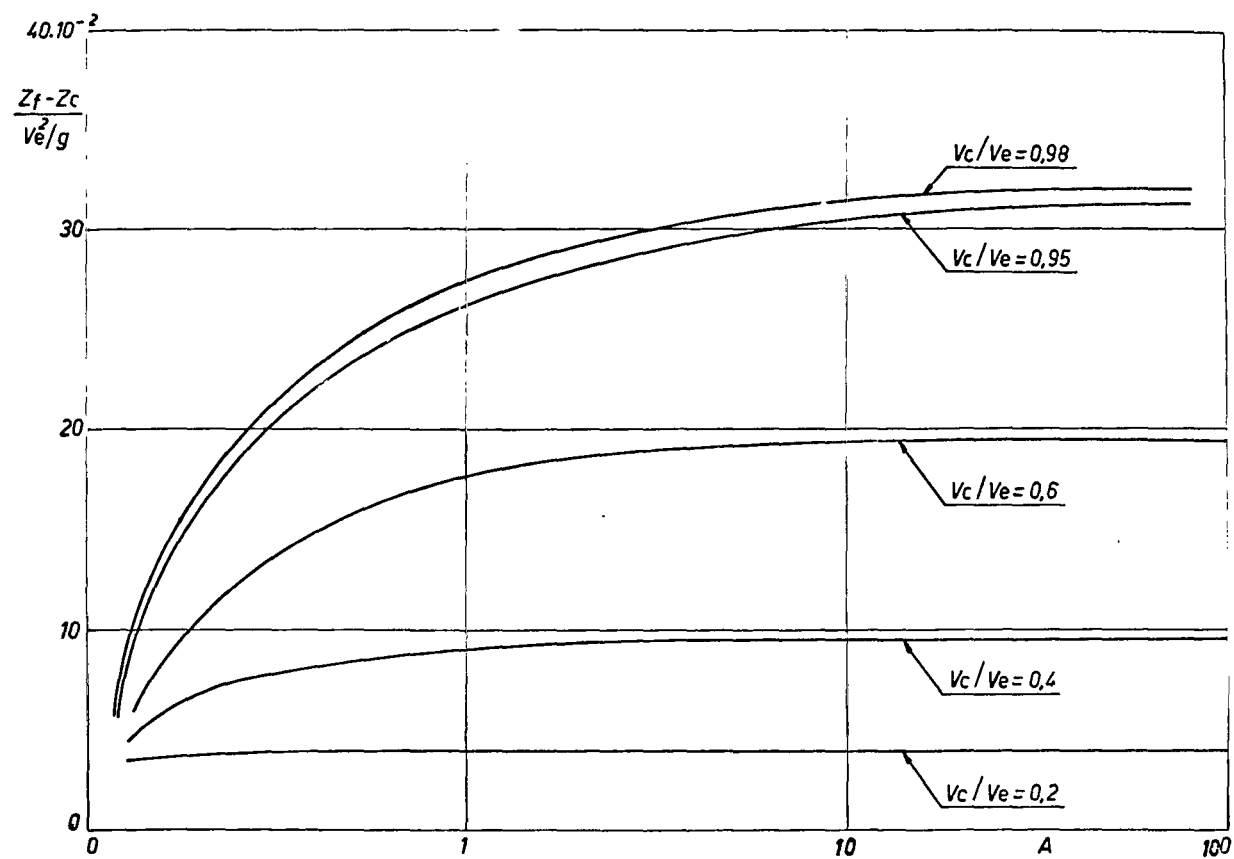
Figure 10
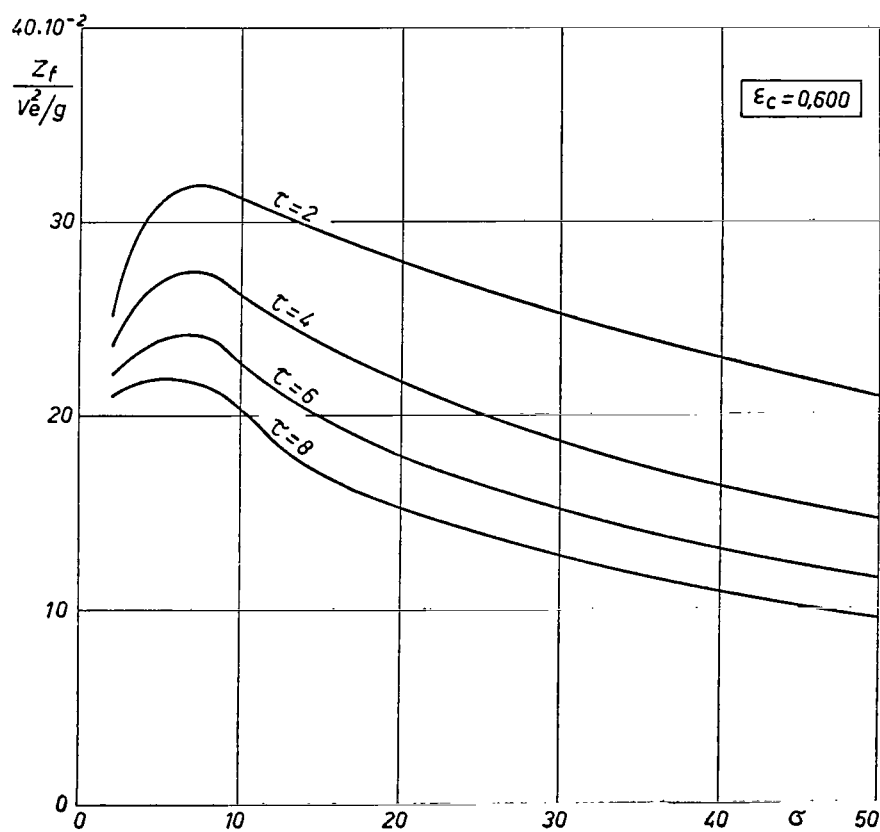
Figure 11
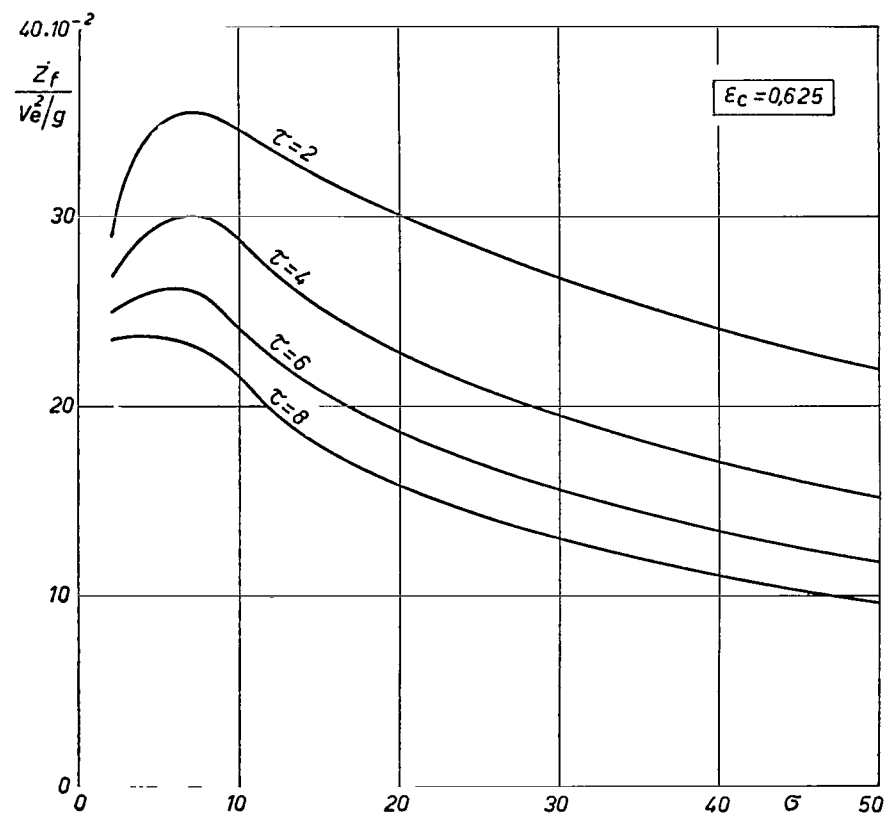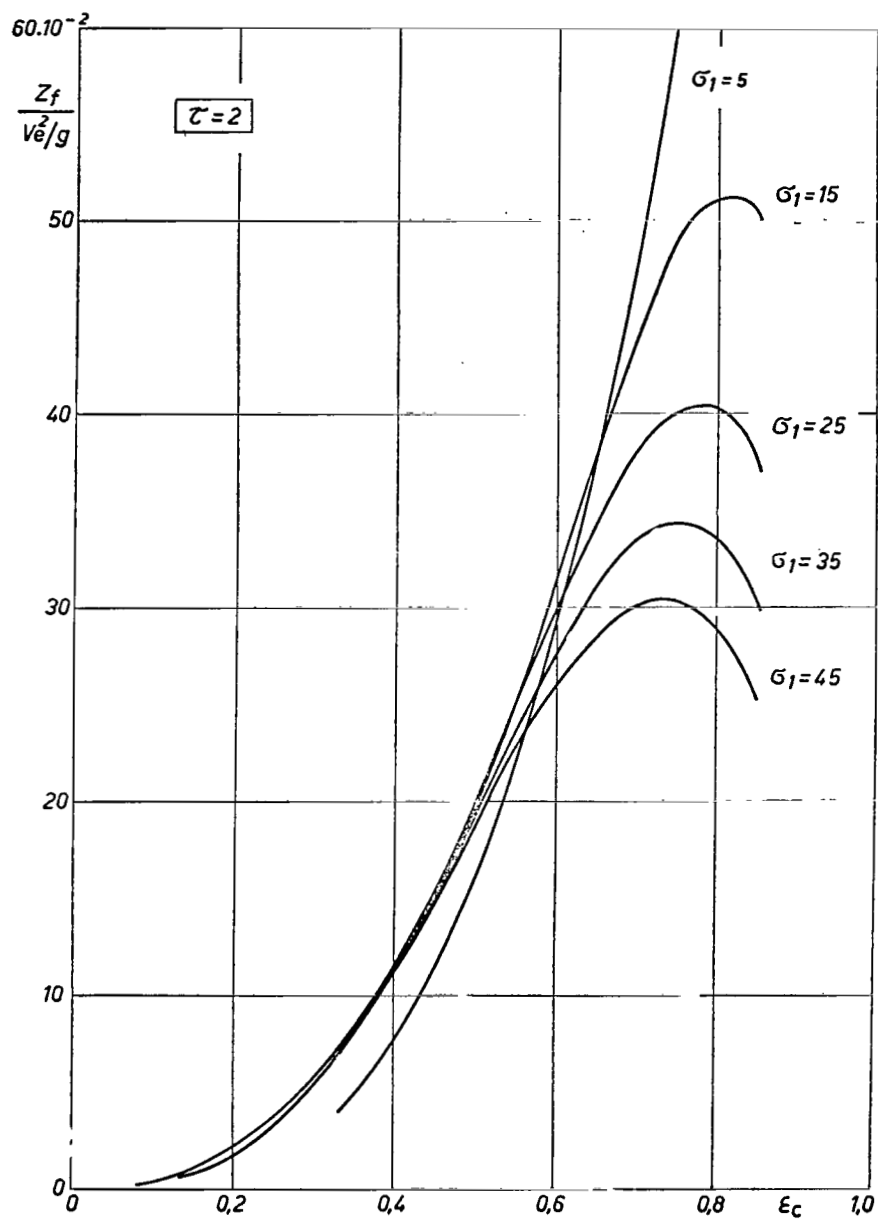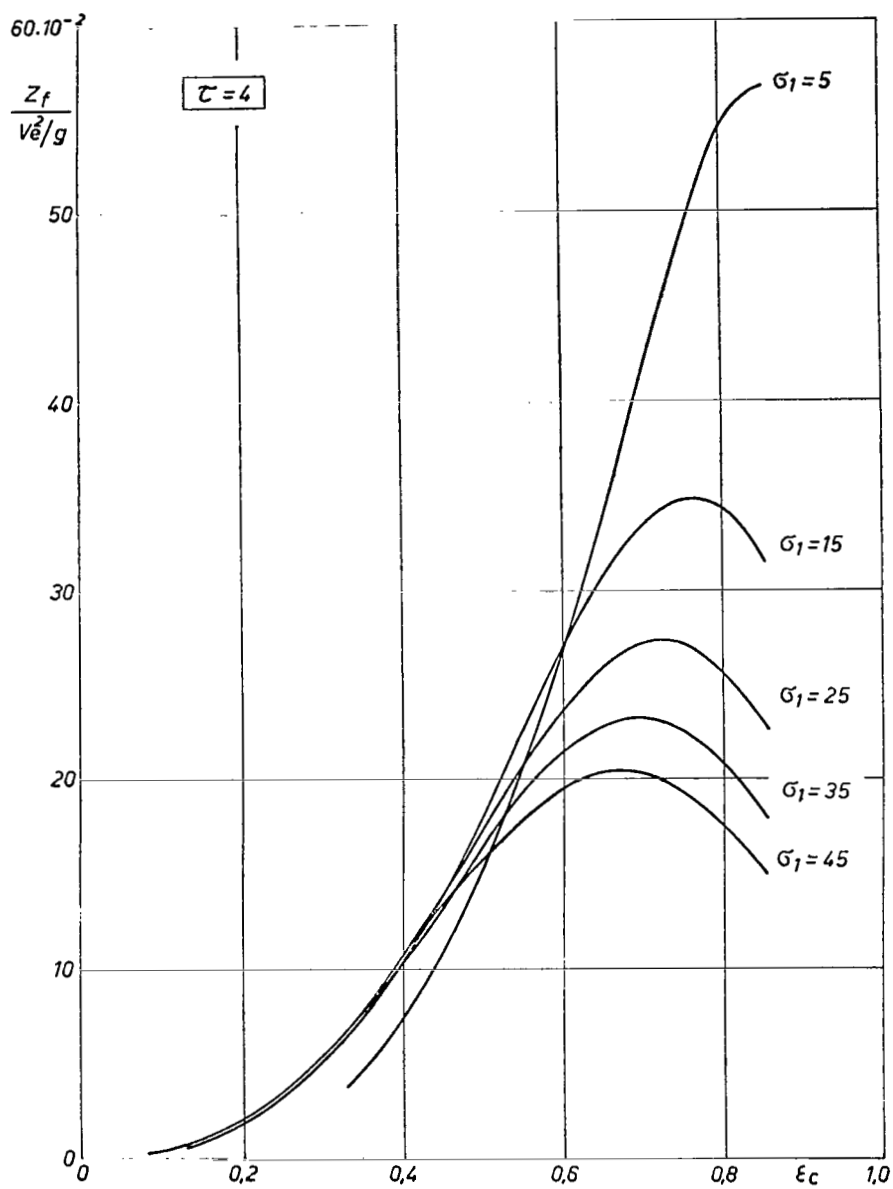
Figure 12
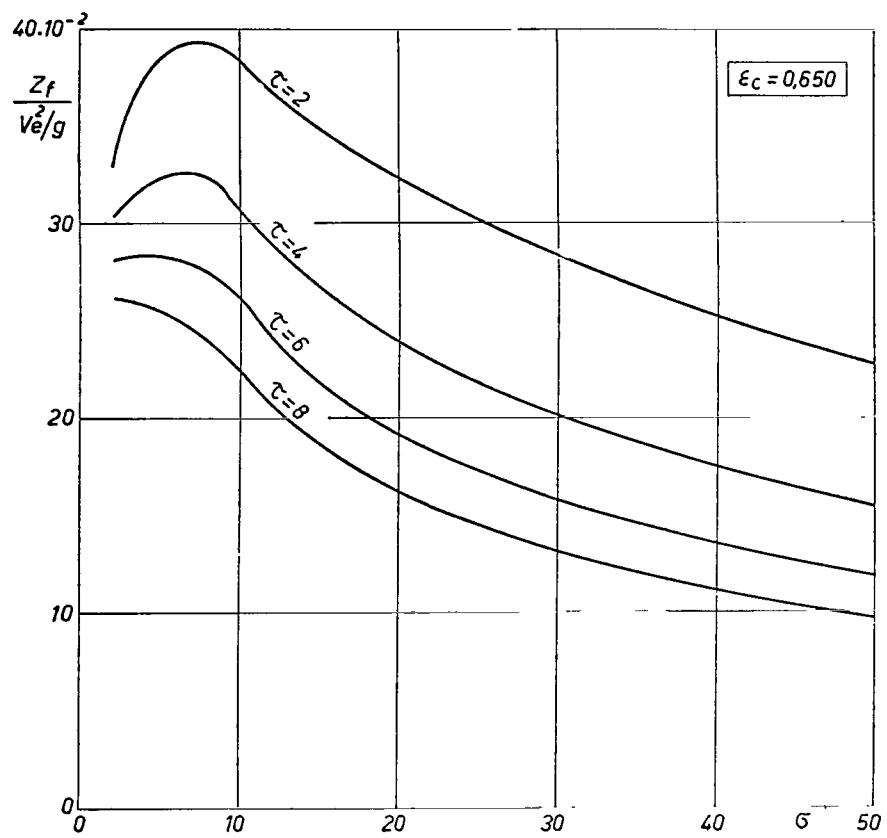
Figure 13
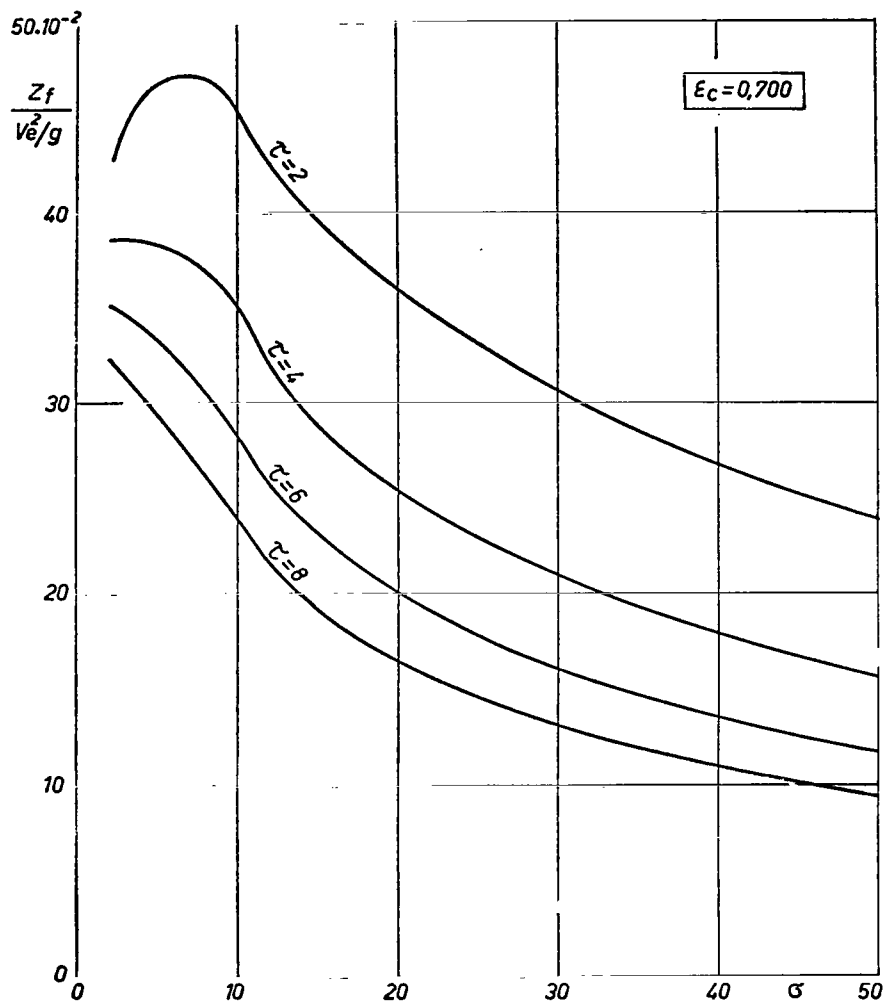
Figure 14
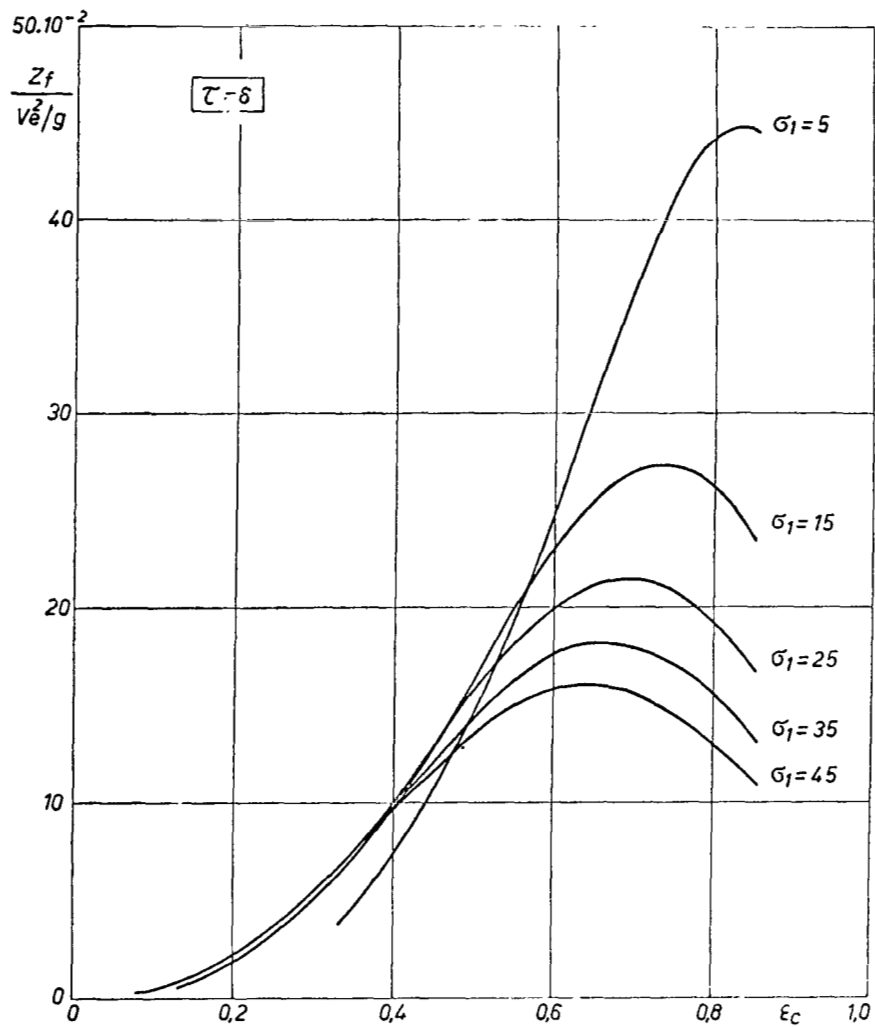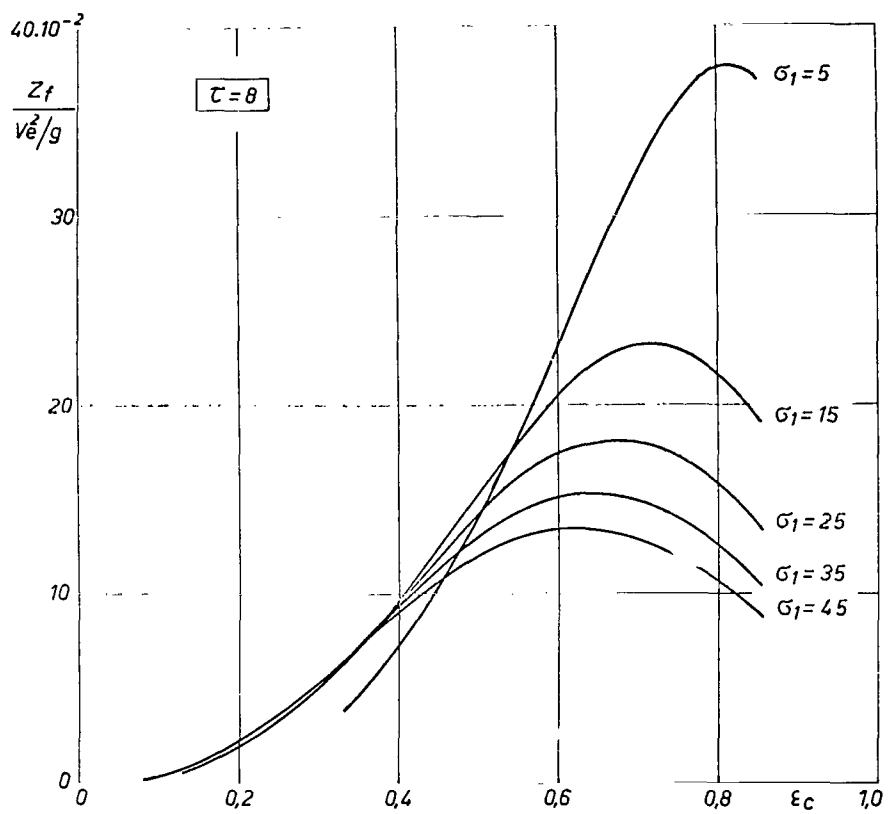
Figure 15

Figure 16

Figure 17

Figure 18

REFERENCES

1. Casci, C.: On the Resolutions of Differential Equations for Missile Motion. Physico Mathematical Studies from Ghisleriani Studies.
2. Casci, C.: On the Possibility of Missiles in Vertical Flight. Deductions of Fundamental Parameters of the Missile Design from the Characteristics of Its Position. Presented at the 5th Astronautical Congress, Copenhagen, 1955.
3. Chin, S. S.: Missile Configuration Design.
4. Seifert, H. S.: Space Technology.
5. Boyd, R. L. F. and M. J. Seaton: Rocket Exploration of the Upper Atmosphere.
6. Davis, L., J. Fallin and L. Blitzer: The Exterior Ballistics of Rockets; Aerodynamic Forces.